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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZN 86/78 JULI

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# A CONVEXITY THEOREM FOR SEQUENCES

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A CONVEXITY THEOREM FOR SEQUENCES

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A convexity theorem for sequences

by

J. van de Lune

ABSTRACT

It is shown that if a sequence is convex then certain derived weighted average sequences are also convex.

KEY WORDS & PHRASES: *convexity, inequalities.*

In [2] it was shown by Ozeki (also see [1; p. 202]) that if a sequence $\{a_n\}_{n=0}^{\infty}$ is convex then also the sequence of corresponding Cesàro means

$$\left\{\frac{a_0 + a_1 + \dots + a_{n-1} + a_n}{n+1}\right\}_{n=0}^{\infty}$$

is convex.

In this note we give a generalization of this result.

<u>THEOREM</u>. *If the sequence* $\{a_n\}_{n=0}^{\infty}$ *is convex, i.e.*

$$a_{n-1} + a_{n+1} \geq 2a_n, \qquad (n = 1,2,3,\dots),$$

*then for any fixed* $k \in \{0,1,2,3,\dots\}$ *the sequence*

$$(*) \qquad \left\{\frac{\binom{k}{k}a_k + \binom{k+1}{k}a_{k+1} + \dots + \binom{n}{k}a_n}{\binom{n+1}{k+1}}\right\}_{n=k}^{\infty}$$

*is also convex.*

<u>PROOF</u>. The convexity of the sequence (*) may be expressed as

$$\frac{\binom{k}{k}a_k + \dots \binom{n}{k}a_n}{\binom{n+1}{k+1}} + \frac{\binom{k}{k}a_k + \dots + \binom{n+2}{k}a_{n+2}}{\binom{n+3}{k+1}} \geq 2\,\frac{\binom{k}{k}a_k + \dots + \binom{n+1}{k}a_{n+1}}{\binom{n+2}{k+1}}$$

where $k \in \{0,1,2,3,\dots\}$ is fixed and $n \geq k$.

After multiplication by the factor

$$(n+3)(n+2)\ \dots\ (n-k+3)(n-k+2)(n-k+1)/(k+1)!$$

we obtain the equivalent inequality

$$(n+2)(n+3)\{\binom{k}{k}a_k + \ldots + \binom{n}{k}a_n\} +$$

$$+ \ (n-k+2)(n-k+1)\{\binom{k}{k}a_k + \ldots + \binom{n+2}{k}a_{n+2}\} +$$

$$- \ 2(n+3)(n-k+1)\{\binom{k}{k}a_k + \ldots + \binom{n+1}{k}a_{n+1}\} \geq 0.$$

For $r = k,k+1,\ldots,n$ the coefficient $c_r$ of the term $a_r$ is

$$c_r = \{(n+2)(n+3) + (n-k+2)(n-k+1) - 2(n+3)(n-k+1)\}\binom{r}{k} =$$

$$= (k+1)(k+2)\binom{r}{k}.$$

The coefficient $c_{n+1}$ of $a_{n+1}$ is

$$c_{n+1} = -2(n+3)(n-k+1)\binom{n+1}{k} + (n-k+2)(n-k+1)\binom{n+1}{k} =$$

$$= -(n-k+1)(n+k+4)\binom{n+1}{k}$$

whereas the coefficient $c_{n+2}$ of $a_{n+2}$ is

$$c_{n+2} = (n-k+1)(n-k+2)\binom{n+2}{k}.$$

Now observe that from the convexity of $\{a_n\}_{n=0}^{\infty}$ it follows that for $r = k,k+1,\ldots,n$

$$(r-k+1)(r-k+2)\binom{r+2}{k}a_r + (r-k+1)(r-k+2)\binom{r+2}{k}a_{r+2} +$$

$$- \ 2(r-k+1)(r-k+2)\binom{r+2}{k}a_{r+1} \geq 0.$$

Add these inequalities and observe that (in the sum) for $r = k,k+1,\ldots,n$ the coefficient $c_r^*$ of $a_r$ is

$$c_r^* = (r-k+1)(r-k+2)\binom{r+2}{k} + (r-k-1)(r-k)\binom{r}{k} - 2(r-k)(r-k+1)\binom{r+1}{k}$$

$$= (k+1)(k+2)\left\{\binom{r+2}{k+2} + \binom{r}{k+2} - 2\binom{r+1}{k+2}\right\}.$$

Since, as one may verify

$$\binom{r+2}{k+2} + \binom{r}{k+2} - 2\binom{r+1}{k+2} = \binom{r}{k}$$

we find that $c_r^* = c_r$ for $r = k, k+1, \ldots, n$.
Assigning the obvious meaning to $c_{n+1}^*$ and $c_{n+2}^*$ we have

$$c_{n+1}^* = (n-k)(n-k+1)\binom{n+1}{k} - 2(n-k+1)(n-k+2)\binom{n+2}{k}$$

and

$$c_{n+2}^* = (n-k+1)(n-k+2)\binom{n+2}{k}.$$

Since

$$c_{n+2}^* = c_{n+2}$$

our proof will be complete if we can show that

$$c_{n+1}^* = c_{n+1}$$

or, equivalently (after division by n-k+1)

$$(n-k)\binom{n+1}{k} - 2(n-k+2)\binom{n+2}{k} = -(n+k+4)\binom{n+1}{k}.$$

This last inequality simplifies to

$$(n+2)\binom{n+1}{k} = (n-k+2)\binom{n+2}{k}$$

or, equivalently (after multiplication by k!)

$$(n+2)(n+1)n(n-1)\dots(n-k+2) = (n-k+2)(n+2)(n+1)n(n-1)\dots(n-k+3).$$

Since this equality holds true indeed, our proof is complete.

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[1] MITRINOVIĆ, D.S., *Analytic inequalities*, Springer, (1970).

[2] OZEKI, N., *On some inequalities*, J. College of Arts and Sci. Chiba Univ., 4 No.3 (1965) pp. 211-214 (in Japanese).